Boolean Functions and Expressions

Definition: The Boolean functions F and G of n variables are **equal** if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B.

Two different Boolean expressions that represent the same function are called **equivalent**.

For example, the Boolean expressions xy, $xy + 0$, and $xy \cdot 1$ are equivalent.

Boolean Functions and Expressions

The **complement** of the Boolean function F is the function –F, where $-F(b_1, b_2, \ldots, b_n) = -(F(b_1, b_2, \ldots, b_n))$.

Let F and G be Boolean functions of degree n. The **Boolean sum F+G** and **Boolean product FG** are then defined by

$(F + G)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n) + G(b_1, b_2, \ldots, b_n)$

$(FG)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n)\, G(b_1, b_2, \ldots, b_n)$

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 1 are there?

Solution: There are four of them, F_1, F_2, F_3, and F_4:

x	F_1	F_2	F_3	F_4
0	0	0	1	1
1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 2 are there?

Solution: There are 16 of them, F_1, F_2, …, F_{16}:

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree n are there?

Solution:

There are 2^n different n-tuples of 0s and 1s.

A Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples.

Therefore, there are 2^{2^n} different Boolean functions.

Duality

There are useful identities of Boolean expressions that can help us to transform an expression A into an equivalent expression B (see Table 5 on page 705 in the textbook).

We can derive additional identities with the help of the **dual** of a Boolean expression.

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Duality

Examples:

The dual of $x(y + z)$ is $x + yz$.

The dual of $-x \cdot 1 + (-y + z)$ is $(-x + 0)((-y)z)$.

The **dual of a Boolean function F** represented by a Boolean expression is the function represented by the dual of this expression.

This dual function, denoted by F^d, **does not depend** on the particular Boolean expression used to represent F.

Duality

Therefore, an identity between functions represented by Boolean expressions **remains valid** when the duals of both sides of the identity are taken.

We can use this fact, called the **duality principle**, to derive new identities.

For example, consider the absorption law $x(x + y) = x$.

By taking the duals of both sides of this identity, we obtain the equation $x + xy = x$, which is also an identity (and also called an absorption law).

Definition of a Boolean Algebra

All the properties of Boolean functions and expressions that we have discovered also apply to **other mathematical structures** such as propositions and sets and the operations defined on them.

If we can show that a particular structure is a Boolean algebra, then we know that all results established about Boolean algebras apply to this structure.

For this purpose, we need an **abstract definition** of a Boolean algebra.

Definition of a Boolean Algebra

Definition: A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation – such that the following properties hold for all x, y, and z in B:

$x \vee 0 = x$ and $x \wedge 1 = x$ (identity laws)

$x \vee (-x) = 1$ and $x \wedge (-x) = 0$ (domination laws)

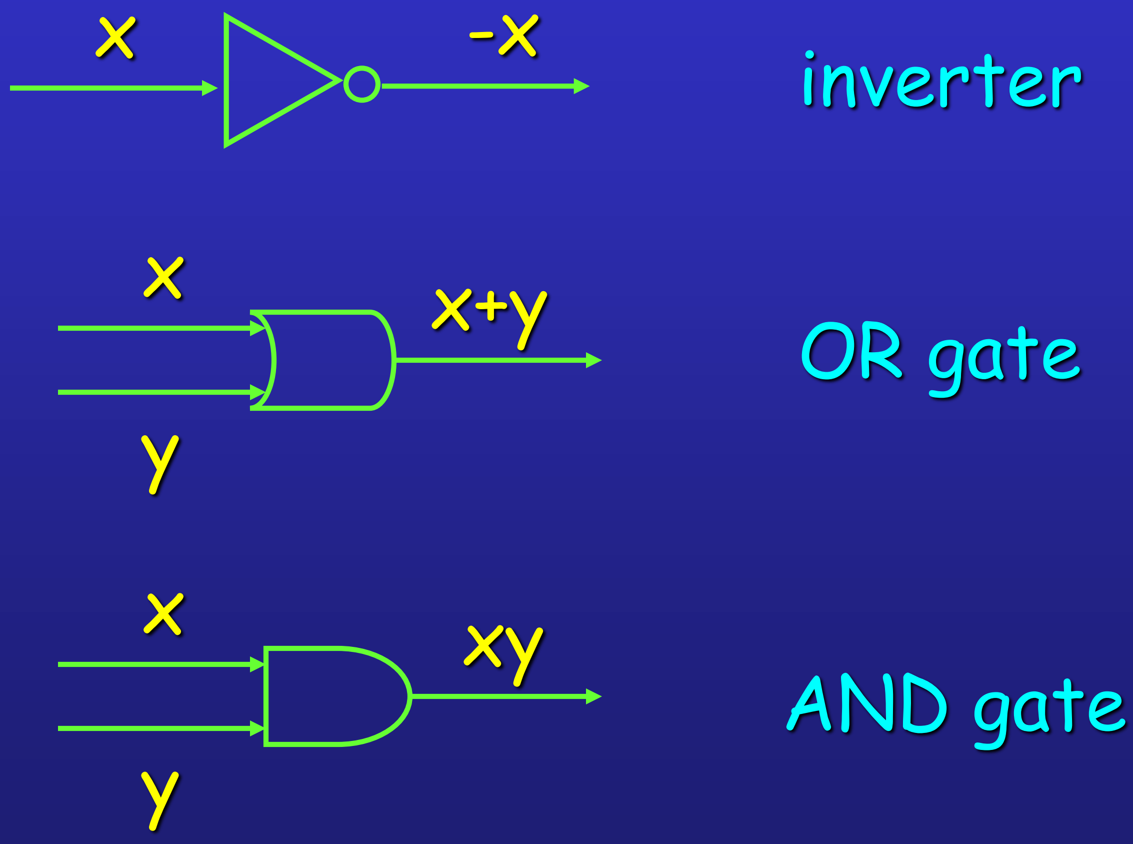
$(x \vee y) \vee z = x \vee (y \vee z)$ and
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and (associative laws)

$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutative laws)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributive laws)

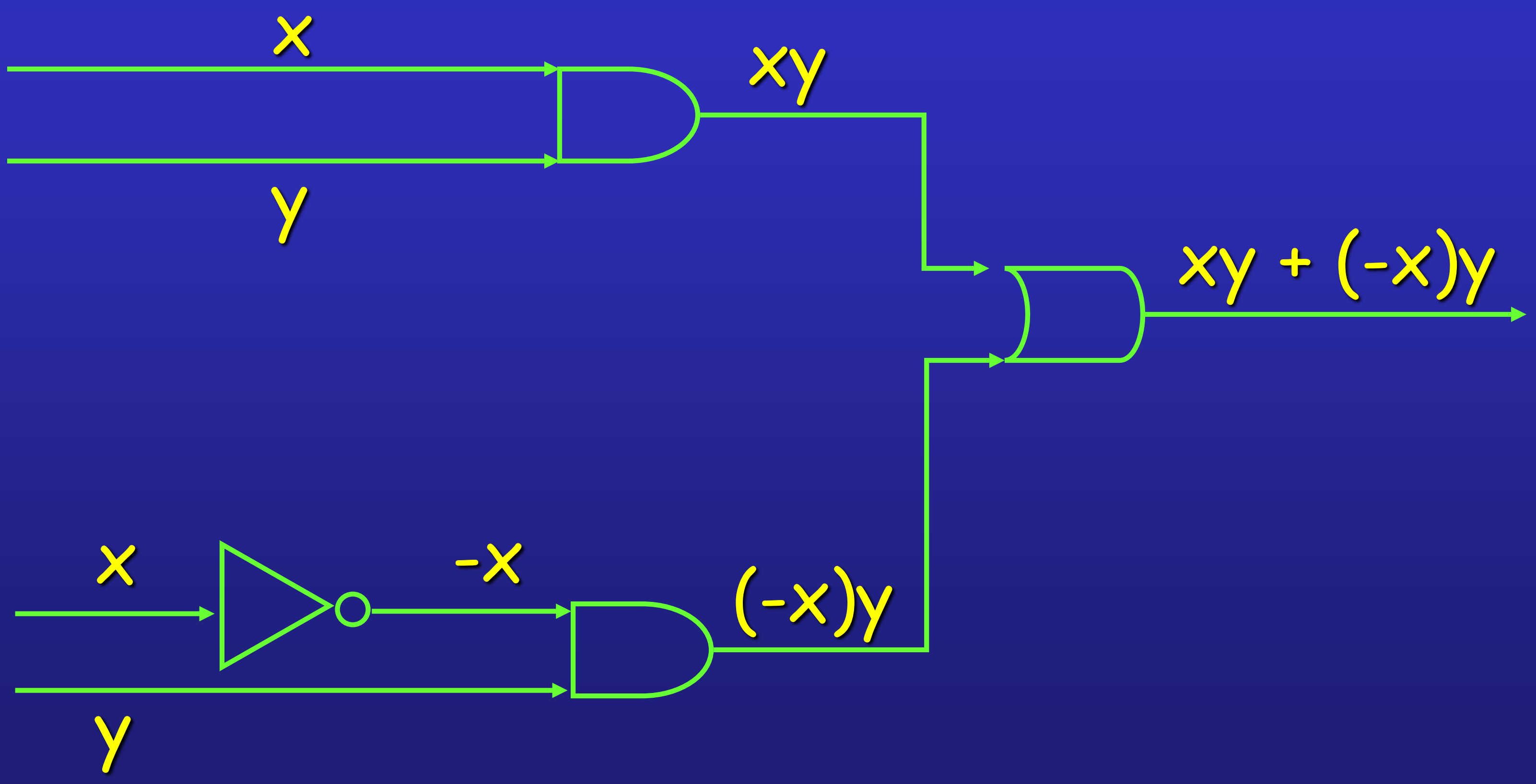
Logic Gates

Electronic circuits consist of so-called gates.
There are three basic types of gates:



Logic Gates

Example: How can we build a circuit that computes the function xy + (-x)y ?



Logic, Sets, and Boolean Algebra

Logic	Set	Boolean Algebra
False	$\varnothing$	0
True	U	1
$A \wedge B$	$A \cap B$	$A \cdot B$
$A \vee B$	$A \cup B$	$A+B$
$\neg A$	A^c	$\overline{A}$

Compare the equivalence laws of them